

# Sufficient conditions placed on initial system-environment states for positive maps

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A system interacting with its environment will give rise to a quantum evolution. After tracing over the environment the net evolution of the system can be described by a linear Hermitian map. It has recently been shown that a necessary and sufficient condition for this evolution to be completely positive is for the initial state to have vanishing quantum discord. In this paper, we provide a sufficient condition for the map to be positive with respect to the initial system-environment correlation. This could lead to ways in which to identify positive but not completely positive maps. Illustrative examples and suggestive procedures are also provided.

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## I. INTRODUCTION

Any quantum system will inevitably interact with its environment in some way. Since the environment is generally not available to us, it is the system alone which is typically observed or measured. As a result the system is open and does not evolve in a unitary fashion. Dynamical maps were proposed to describe the state of a system [1]. The maps can be classified as being either positive or non-positive, with the positive maps including completely positive (CP) maps.

Suppose the system  $A$  interacts with an environment  $E$ . After an evolution determined by the standard quantum-mechanical prescription, its density matrix at a given time  $t$  will reduce to

$$\begin{aligned}\rho_A(t) &= \text{Tr}_E[\rho_{AE}(t)] \\ &= \text{Tr}_E[U_{AE}(t)\rho_{AE}(0)U_{AE}(t)^\dagger] \\ &\equiv \mathcal{S}[\rho_A(0)],\end{aligned}\quad (1)$$

where  $U_{AE}(t)$  is a unitary matrix determined by the joint system-environment Hamiltonian,  $\mathcal{S}$  is the induced map, and  $\rho_A(0) = \text{Tr}_E\rho_{AE}(0)$ . In recent years there has been an extensive investigation regarding the conditions imposed on the initial state of a composite system which lead to either positive or CP maps [2–6].

It is well known that if the initial state  $\rho_{AE}(0)$  is of a simple product form, i.e.,  $\rho_{AE}(0) = \rho_A \otimes |0\rangle_E\langle 0|$ , the resulting map  $\mathcal{S}$  is CP [7]. Simply separable states are not the only ones whose evolution can be described by a CP map [4], the general class consists of those states with vanishing quantum discord (VQD) [8]. It has been shown that such a quantum dynamical process (1) always leads to a linear Hermitian map  $S$ , and for arbitrary  $U_{AE}(t)$  the initial state with VQD is not only sufficient [4] but also necessary for CP maps [6]. Positive but not CP maps play an important role in detecting entanglement of quantum states [9, 10]. Using matrix algebras, some positive maps were constructed in Ref. [11]. However, with the exception of the CP maps, we know little about the condition(s) which must be imposed on an initial state so that the subsequent evolution is a positive map for arbitrary  $U_{AE}$ .

In this paper, we will give a sufficient condition for the maps  $\mathcal{S}$  (1) to be positive with respect to the initial composite state and conjecture this condition is necessary as well. This result, together with that of [4, 6] may provide an efficient way of finding some positive but not CP maps.

## II. SUFFICIENT CONDITIONS FOR POSITIVITY

A separable quantum state  $\rho_{AE}$  on  $\mathcal{H}_A \otimes \mathcal{H}_E$  with  $d \otimes f$  dimensions can be expressed as a convex combination of product states [12], i.e., in the form

$$\rho_{AE} = \sum_i p_i \rho_A^{(i)} \otimes \rho_E^{(i)}, \quad (2)$$

with nonnegative  $p_i$  satisfying  $\sum_i p_i = 1$ . The state (2) can be rewritten as

$$\rho_{AE} = \sum_{kl} \Gamma_{kl} |k\rangle\langle l| \otimes \psi_{kl}, \quad (3)$$

where  $\{|k\rangle\}_{k=1}^d$  represents an orthonormal basis for the Hilbert space of system  $A$ ,  $\mathcal{H}_A$ , and  $\{\psi_{kl}\}_{k,l=1}^d : \mathcal{H}_E \mapsto \mathcal{H}_E$  are normalized such that if  $\text{Tr}[\psi_{kl}] \neq 0$  then  $\text{Tr}[\psi_{kl}] = 1$ . The reduced density matrix of the system  $A$  is

$$\rho_A = \sum_{(k,l) \in \mathcal{C}} \Gamma_{kl} |k\rangle\langle l|, \quad (4)$$

where  $\mathcal{C} \equiv \{(k, l) | \text{Tr}[\psi_{kl}] = 1\}$ . The special-linear (SL) class of states [6] is defined such that  $\text{Tr}[\psi_{kl}] = 1$  or  $\psi_{kl} = 0, \forall k, l$ . Furthermore, for the SL class we have  $\Gamma_{kl} \neq 0$  for  $\psi_{kl}$  with  $\text{Tr}[\psi_{kl}] = 1$  or  $\psi_{kl} \neq 0$ , while  $\Gamma_{kl} = 0$  for  $\psi_{kl} = 0$ .

We denote the elements of component  $\rho_A^{(i)}$  in (2) by  $\mathcal{E}_{kl}^{(i)}$ , i.e.,  $\rho_A^{(i)} = \sum_{kl} \mathcal{E}_{kl}^{(i)} |k\rangle\langle l|$ , such that for the separable SL class the bath operator  $\psi_{kl}$  can be written

$$\psi_{kl} = \sum_i \frac{p_i \mathcal{E}_{kl}^{(i)}}{\Gamma_{kl}} \rho_E^{(i)}, \quad (5)$$

with  $\Gamma_{kl} = \Sigma_i p_i \mathcal{E}_{kl}^{(i)}$  for  $\Gamma_{kl} \neq 0$ . Rewriting the dynamical map (1) we have

$$\mathcal{S}[|k\rangle\langle l|] = \text{Tr}_E[U_{AE}(t)(|k\rangle\langle l| \otimes \psi_{kl})U_{AE}(t)^\dagger], \quad (6)$$

for the SL class. On the other hand, there is a shift term which is independent of  $\rho_A$  for the non-SL class [6]. If  $\psi_{kl} = 0$ ,  $\mathcal{S}[|k\rangle\langle l|] = 0$ , i.e., the corresponding basis  $|k\rangle\langle l|$  has no contribution to the resulting state. In what follows we will assume that the system and bath are initially in a separable SL class state.

In order to find the condition under which the map  $\mathcal{S}$  is positive, we need to apply  $\mathcal{S}$  to an arbitrary  $d \times d$  density matrix

$$\rho'_A = \sum_{kl} \Gamma'_{kl} |k\rangle\langle l|, \quad (7)$$

to see whether the resulting matrix

$$\mathcal{S}[\rho'_A] = \sum_{kl} \text{Tr}_E[U_{AE}(t)(\Gamma'_{kl}|k\rangle\langle l| \otimes \psi_{kl})U_{AE}(t)^\dagger] \quad (8)$$

is positive or not. Let us define a set of matrices as

$$\varrho_A^{(i)} = \sum_{kl} \frac{\Gamma'_{kl} \mathcal{E}_{kl}^{(i)}}{\Gamma_{kl}} |k\rangle\langle l| \equiv \sum_{kl} \Gamma'_{kl} \Gamma_{kl}^{(i)} |k\rangle\langle l|, \quad (9)$$

with  $\Gamma_{kl}^{(i)} = \mathcal{E}_{kl}^{(i)} / \Gamma_{kl}$ . Using (5) and (9) we can reexpress (8) as

$$\begin{aligned} \mathcal{S}[\rho'_A] &= \sum_{ikl} p_i \text{Tr}_E[U_{AE}(t)(\Gamma'_{kl} \Gamma_{kl}^{(i)} |k\rangle\langle l| \otimes \rho_E^{(i)}) U_{AE}(t)^\dagger] \\ &= \sum_i p_i \text{Tr}_E[U_{AE}(t)(\varrho_A^{(i)} \otimes \rho_E^{(i)}) U_{AE}(t)^\dagger]. \end{aligned} \quad (10)$$

From (10) it is apparent that if the matrix  $\varrho_A^{(i)} \geq 0, \forall i$ ,  $\mathcal{S}[\rho'_A]$  will be positive, the sum of positive density matrices is indeed positive. Since  $\rho'_A$  represents an arbitrary density matrix, having  $\varrho_A^{(i)} \geq 0$  for  $\forall i$  implies the mapping  $\mathcal{S}$  will be positive as well.

Before proceeding, let us state the following Lemma.

**Lemma 1:** For two positive matrices defined by  $\rho_1 = \sum_{ij} \phi_{ij} |i\rangle\langle j|$  and  $\rho_2 = \sum_{ij} \varphi_{ij} |i\rangle\langle j|$ , there exists an unnormalized matrix  $\rho$  such that

$$\rho \equiv \sum_{ij} \phi_{ij} \varphi_{ij} |i\rangle\langle j| \geq 0. \quad (11)$$

**Proof:** This proof is straightforward. Since  $\rho_1 \otimes \rho_2$  is nonnegative, its principal submatrix is also nonnegative. It is clear that the matrix  $\rho$  is a principal submatrix of  $\rho_1 \otimes \rho_2$ . Thus (11) follows.

Using this Lemma, the comparison between (9) and (11) now provides us with a condition for the  $\varrho_A^{(i)}$  to be positive. If the re-scaled matrices

$$\varrho_R^{(i)} \equiv \sum_{kl} \Gamma_{kl}^{(i)} |k\rangle\langle l|, \forall i \quad (12)$$

are all nonnegative, then  $\varrho_A^{(i)} \geq 0, \forall i$ , and thus  $\mathcal{S}[\rho'_A] \geq 0$ . Note that the equation above is only dependent on the initial state. Therefore, we can draw the conclusion that for an arbitrary  $U_{AE}$ , the map  $\mathcal{S}$  defined by (1) is positive if the initial system-bath state  $\rho_{AE}$  is in the separable SL class and all of its re-scaled matrices (12) are nonnegative.

It is not difficult to verify that the conclusion above may be reformulated in the following way, which constitutes the main result of this paper.

**Theorem:** For an arbitrary  $U_{AE}$ , the map  $\mathcal{S}$  defined by (1) is positive if the initial system-bath SL state  $\rho_{AE}$  in  $d \otimes f$  dimensions is of the unentangled form (2) and the component  $\varrho_A^{(i)}$  can be written as

$$\varrho_A^{(i)} = \Pi_i^{(d_i)} \rho_A^{(i)} \Pi_i^{(d_i)}, \quad (13)$$

where  $\{\Pi_i^{(d_i)}\}$  are  $d_i$ -dimensional projectors onto  $\rho_A^{(i)}$  and  $\sum_i \Pi_i^{(d_i)} = I_d$  with  $\Sigma_i d_i = d$ .

### A. Example

For an intuitive picture of the theorem, an illustration is given by the following example. Consider the separable initial  $4 \otimes f$  state

$$\rho_{AE} = p_1 \rho_A^{(1)} \otimes \rho_E^{(1)} + p_2 \rho_A^{(2)} \otimes \rho_E^{(2)}, \quad (14)$$

where  $\rho_A^{(i)}$  is of the form in the computational basis  $\{|0\rangle|1\rangle|2\rangle|3\rangle\}$

$$\rho_A^{(1)} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv \sum_{k,l=1}^4 \mathcal{E}_{kl}^{(1)} |k\rangle\langle l|, \quad (15)$$

and

$$\rho_A^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} \equiv \sum_{k,l=1}^4 \mathcal{E}_{kl}^{(2)} |k\rangle\langle l|.$$

It is easy to check that  $\rho_{AE}$  is a SL state and the reduced state of subsystem A is

$$\rho_A = \begin{pmatrix} p_1 a & p_1 b & 0 & 0 \\ p_1 c & p_1 d & 0 & 0 \\ 0 & 0 & p_2 e & p_2 f \\ 0 & 0 & p_2 g & p_2 h \end{pmatrix} \equiv \sum_{k,l=1}^4 \Gamma_{kl} |k\rangle\langle l|.$$

According to (9) and (12) we can have the re-scaled matrices as follows

$$\varrho_R^{(1)} = \sum_{k,l=1}^4 \Gamma_{kl}^{(1)} |k\rangle\langle l| = \begin{pmatrix} \frac{1}{p_1} & \frac{1}{p_1} & 0 & 0 \\ \frac{1}{p_1} & \frac{1}{p_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\varrho_R^{(2)} = \sum_{k,l=1}^4 \Gamma_{kl}^{(2)} |k\rangle\langle l| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{p_2} & \frac{1}{p_2} \\ 0 & 0 & \frac{1}{p_2} & \frac{1}{p_2} \end{pmatrix}.$$

The two re-scaled matrices above are obviously nonnegative such that the map (1) resulting from the initial state (14) is positive for arbitrary  $U_{AE}$ .

## B. Discussion

Note that if all  $d_i = 1$  in the theorem, then the state has a VQD and the map must be a CP map [6]. (For example if the  $\rho_A^{(i)}$  in Eq. (14) in the example are 1-D projectors.) However, if the state Eq. (2) does not have VQD it may be possible to find  $U_{AE}$  such that the map is not CP. Therefore, the theorem could provide us a way to search for positive but not CP maps by varying  $U_{AE}$ . Since the set of CP maps is a subset of the positive maps, the restriction to the initial states with respect to positive maps is relaxed, compared to one for the CP maps.

It could be the case that the sufficient condition given in the theorem for positive maps is necessary as well. However, it is predicted that one shall encounter more challenge giving a complete proof and thus it deserves further investigation. In the following, we provide an explicit example showing that an initial entangled composite state can lead to a non-positive map by using the analysis of [4, 6].

Consider the initial state in the entangled form

$$\rho_{AE} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (16)$$

Since the state does not belong to the SL class, the resulting map is not necessarily positive. Indeed, the following computation verifies this fact.

As shown by [13], the map  $\mathcal{S}(t)$  induced from (16) assumes an affine form

$$\mathcal{S}(t)[\rho_E(0)] = \mathcal{S}_{SL}(t)[\rho_E(0)] + \mathcal{S}_{NSL}(t), \quad (17)$$

where  $\mathcal{S}_{SL}(t)$  depends on  $\rho_E(0)$  while  $\mathcal{S}_{NSL}(t)$  does not. (For more details on  $\mathcal{S}_{SL}(t)$ , see [13].) Next, we try to apply the map  $\mathcal{S}(t)$  to a particular pure-state density matrix

$$\rho' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (18)$$

Clearly, this matrix is positive semi-definite. From (8)

$$\mathcal{S}_{SL}(t)[\rho'] = \frac{1}{2}\rho'. \quad (19)$$

For the state (16), the shift term  $\mathcal{S}_{NSL}(t)$  in (17) has the form [6]

$$\mathcal{S}_{NSL}(t) = \sum_{kl \in \{01, 10\}} \Gamma_{kl} \text{Tr}_E[U_{AE}(t)|k\rangle\langle l| \otimes \psi_{kl} U_{AE}(t)^\dagger], \quad (20)$$

where  $\Gamma_{01} = \Gamma_{10} = \frac{1}{2}$ , and

$$\psi_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \psi_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (21)$$

Now we choose  $U_{AE}(t)$  as a CNOT gate, i.e.,

$$U_{AE}(t) = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10|, \quad (22)$$

such that we obtain resulting matrix

$$\begin{aligned} \mathcal{S}(t)[\rho'] &= \frac{1}{2}\rho' + \mathcal{S}_{NSL}(t) \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

which is negative. Therefore, we conclude that the map of the state (16) is not always positive for all  $U_{AE}$ .

## III. CONCLUSION

In conclusion, we have obtained a sufficient condition for positive maps with respect to a given initial system-environment state. The positive but not CP maps are important for identifying the entanglement of quantum states, and our result may provide an efficient way of constructing such maps. This can be performed as follows. First, we choose the initial state satisfying the theorem above. Second, we exclude the states with the VQD. Finally, we try a variety of unitary transformations until the desired maps are found. Here, we need the method presented in [5] to determine if the map is CP or not. We leave as an important open problem whether the condition is necessary as well, or which condition(s) are necessary. This would provide an improved method for searching for positive but not CP maps.

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